Fundamental Structures of M(brane) Theory

Jens Hoppe

Department of Mathematics, Royal Institute of Technology
100 44 Stockholm, Sweden

Abstract

A dynamical symmetry, as well as special diffeomorphism algebras generalizing the Witt-Virasoro algebra, related to Poincaré-invariance and crucial with regard to quantisation, questions of integrability, and M(atrix) theory, are found to exist in the theory of relativistic extended objects of any dimension.

The simplicity of classical string theory, and decades of presenting it from one and the same point of view, have made it difficult to realize some of the central features of relativistic extended objects (described below in the light cone gauge), namely:

- relativistic invariance implying the existence of a dynamical symmetry (irrespective of the dimension of the extended object), and
- the Virasoro algebra being just the simplest example of certain (extended) infinite-dimensional diffeomorphism algebras reappearing, after gauge fixing (and on the constrained phase space), in the reconstruction of x^-.

For the purpose of this note, I will restrict myself to the purely bosonic theory [1, 2], i.e. (analogous results for the supersymmetrized theory will easily follow)

$$H = \frac{1}{2\eta}\int_{\Sigma_0} \frac{\vec{p}^2 + g}{\rho}\, d^M\varphi = H[\vec{x}, \vec{p}; \eta, \zeta_0] = \int \mathcal{H}\, d^M\varphi, \tag{1}$$

$$g = \det\left(\frac{\partial \vec{x}}{\partial \varphi^a}\cdot\frac{\partial \vec{x}}{\partial \varphi^b}\right)_{a,b=1,\ldots,M},$$

with $\rho(\varphi)$ being a non-dynamical density of weight one (i.e. $\int_{\Sigma_0} \rho(\varphi)\, d^M\varphi = 1$), x_i and p_j $(i, j = 1, \dots, d = D-2)$ canonically conjugate fields satisfying

$$\int f^a\, \vec{p} \cdot \partial_a \vec{x}\, d^M\varphi = 0 \quad \text{whenever} \quad \nabla_a f^a = 0 \tag{2}$$

for the consistency of

$$\eta \partial_a \zeta = \frac{\vec{p}}{\rho} \cdot \partial_a \vec{x} \tag{3}$$

which, together with

$$2\eta^2 \dot{\zeta} = \frac{\vec{p}^{\,2} + g}{\rho^2} \tag{4}$$

(that actually can also be thought of as defining η and ρ in terms of the initial parametrized shape, and the velocity, of the time-dependent M-dimensional extended object moving in D-dimensional Minkowski space), determine ζ (usually called x^-) up to $\zeta_0 = \int \zeta \rho\, d^M\varphi$; the time independent positive degree of freedom η (usually called P^+) is canonically conjugate to $-\zeta_0$.

In the mid-eighties, Goldstone (when proving that the above description is fully Poincaré-invariant [2]) solved (3) (assuming (2)) in the form

$$\zeta(\varphi) = \zeta_0 + \frac{1}{\eta} \int G(\varphi, \tilde{\varphi}) \tilde{\nabla}^a \left(\frac{\vec{p}}{\rho} \cdot \tilde{\nabla}_a \vec{x} \right)(\tilde{\varphi})\, \rho(\tilde{\varphi})\, d^M\tilde{\varphi} \tag{5}$$

with (∇_a being the covariant derivative, and Δ the Laplacian on Σ_0)

$$\int G(\varphi, \tilde{\varphi}) \rho(\varphi)\, d^M\varphi = 0, \qquad \Delta_{\tilde{\varphi}} G(\varphi, \tilde{\varphi}) = \frac{\delta(\varphi, \tilde{\varphi})}{\rho(\varphi)} - 1. \tag{6}$$

Later it will turn out to be useful to slightly (though, with regard to a variety of aspects: crucially) rewrite (5) as

$$\zeta(\varphi) = \zeta_0 + \frac{1}{2\eta} \vec{p} \cdot \vec{x} + \frac{1}{2} \int G(\varphi, \tilde{\varphi}) \left(\frac{\vec{p}}{\rho} \cdot \Delta \vec{x} - \vec{x} \cdot \Delta \frac{\vec{p}}{\rho} \right)(\tilde{\varphi})\, \rho(\tilde{\varphi})\, d^M\tilde{\varphi} \tag{7}$$

(splitting $\zeta - \zeta_0$ into parts symmetric resp. antisymmetric with regard to interchanging $\vec{x}$ and $\vec{p}$, and involving only the invariant Laplace operator). I can now present the two key features that I recently found:

Dynamical Symmetry

When separating the zero-modes

$$\zeta_0, \qquad \eta, \qquad X_i = \int x_i \rho\, d^M\varphi, \qquad P_i = \int p_i\, d^M\varphi \tag{8}$$

from the internal degrees of freedom,

$$x_{i\alpha} := \int Y_\alpha(\varphi) x_i(\varphi)\rho(\varphi)\, d^M\varphi, \qquad p_{i\alpha} := \int Y_\alpha(\varphi) p_i(\varphi)\, d^M\varphi, \tag{9}$$

— letting $\{Y_\alpha\}_{\alpha=1}^\infty$ be a (together with $Y_0 = 1$) complete orthonormal set of eigenfunctions on Σ_0 (conveniently chosen as eigenfunctions of Δ),

$$\int Y_\alpha Y_\beta \rho\, d^M\varphi = \delta_{\alpha\beta}, \;\; \sum_{\alpha=1}^\infty Y_\alpha(\varphi) Y_\alpha(\tilde\varphi) = \frac{\delta(\varphi,\tilde\varphi)}{\rho(\varphi)} - 1, \;\; \Delta Y_\alpha = -\mu_\alpha Y_\alpha \tag{10}$$

— the Lorentz-invariance of the theory, in particular implying that

$$M_{i-} := \int (x_i \mathcal{H} - \zeta p_i)\, d^M\varphi \tag{11}$$

satisfies

$$\{M_{i-}, M_{j-}\} = 0, \tag{12}$$

necessitates that the purely internal contributions

$$2\eta\mathbb{M}_{i-} := \int (x_i \tilde{\mathcal{H}} - \tilde\zeta p_i)\, d^M\varphi = x_{i\alpha}\tilde{\mathcal{H}}_\alpha - \tilde\zeta_\alpha p_{i\alpha}, \tag{13}$$

with

$$\tilde{\mathcal{H}}_\alpha := \vec p_\beta \cdot \vec p_\gamma \int Y_\alpha Y_\beta Y_\gamma \rho\, d^M\varphi + \int Y_\alpha \frac{g}{\rho}\, d^M\varphi =: \vec p_\beta \cdot \vec p_\gamma d_{\alpha\beta\gamma} + W_\alpha, \tag{14}$$

$$\tilde\zeta_\alpha := 2\eta(\zeta_\alpha - \vec P \cdot \vec x_\alpha), \qquad \zeta_\alpha := \int Y_\alpha \zeta \rho\, d^M\varphi,$$

satisfy

$$\{\eta\mathbb{M}_{i-}, \eta\mathbb{M}_{j-}\} = \mathbb{M}^2\, \mathbb{M}_{ij}, \qquad i,j = 1,\ldots,d, \tag{15}$$

where

$$\mathbb{M}_{ij} := x_{i\alpha} p_{j\alpha} - x_{j\alpha} p_{i\alpha} \tag{16}$$

are the generators of internal transverse rotations, and

$$\mathbb{M}^2 = \tilde{\mathcal{H}}_\alpha \tilde{\mathcal{H}}_\alpha = 2\eta H - \vec P^2 \tag{17}$$

is the square of the relativistically invariant 'internal mass', commuting with M_{i-}, M_{ij}, H, $\vec P$, and η, as well as $\eta\zeta_0$ and ηX_i. (15) is a simple (but crucial) consequence of (12), as the parts of M_{i-} that do involve the zero-modes satisfy

$$\left\{ X_i H - \zeta_0 P_i + \frac{\mathbb{M}_{ik}}{\eta} P_k \;,\; X_j H - \zeta_0 P_j + \frac{\mathbb{M}_{jl}}{\eta} P_l \right\} = -\frac{\mathbb{M}^2}{\eta^2} \mathbb{M}_{ij}, \tag{18}$$

which easily follows from $\{H, \mathbb{M}_{ik}\} = 0$, $\{\zeta_0, \eta\} = -1$, $\{X_i, P_j\} = \delta_{ij}$, and

$$\{\mathbb{M}_{ik}, \mathbb{M}_{jl}\} = -\delta_{kj}\mathbb{M}_{il} \pm 3 \text{ more}, \tag{19}$$

and (17). Finally, one checks that

$$\{\eta\mathbb{M}_{i-}, \mathbb{M}_{ij}\} = -\eta\mathbb{M}_{j-}, \tag{20}$$

and that $\mathbb{M}^2$ commutes with $\eta\mathbb{M}_{i-}$ (and $\mathbb{M}_{ij}$).

This sign of integrability / dynamical symmetry ($\mathbb{M}^2$ appearing in the structure constants of a symmetry algebra of itself) should be extremely useful for the further understanding of relativistic extended objects. E.g. if it is possible to promote (15), (19), (20) to commutation relations for corresponding quantum operators (commuting with $\hat{\mathbb{M}}^2$), one may be able to calculate the spectrum of $\hat{\mathbb{M}}^2$ purely algebraically in terms of the Casimirs of the algebra spanned by $L_{ij} := \mathbb{M}_{ij}$ and $L_{i,d+1} := \frac{\eta\mathbb{M}_{i-}}{\sqrt{\mathbb{M}^2}}$, just as in the case of the d-dimensional Hydrogen atom, which is actually *very* close to the relations that I just derived; the difference lying in the explicit a-priori relations between the angular momentum $\mathbb{M}_{ij}$ and the generalized Laplace-Runge-Lenz vector (most likely these relations exist here as well, encoding the dimensionality and topology of the extended object). One way to find them is to understand the interplay of the different diffeomorphism subalgebras involving the totally symmetric structure constants $d_{\alpha\beta\gamma}$ (cp. (14)),

$$e_{\alpha\beta\gamma} := \frac{\mu_\beta - \mu_\gamma}{\mu_\alpha} d_{\alpha\beta\gamma} \tag{21}$$

and

$$g_{\alpha\alpha_1\ldots\alpha_M} := \int_{\Sigma_0} Y_\alpha \epsilon^{a_1\ldots a_M} \frac{\partial Y_{\alpha_1}}{\partial\varphi^{a_1}} \cdots \frac{\partial Y_{\alpha_M}}{\partial\varphi^{a_M}}\, d^M\varphi, \tag{22}$$

— part of which I will now come to.

$\mathscr{L}$-Algebras

To directly verify (15) (just using (13)) is a very instructive, but complicated, calculation; in particular one finds that the modes of ζ (times η) close under Poisson-brackets (on the constrained phase-space, i.e. assuming (2)),

$$\{\eta\zeta_\alpha, \eta\zeta_{\alpha'}\} = f^\epsilon_{\alpha\alpha'}\eta\zeta_\epsilon \tag{23}$$

(whose simplest, $M = 1$, example leads to the Virasoro-algebra). Let me calculate the structure constants and identify the generators as special diffeomorphisms of Σ_0:

Using (3)/(5) and (6)/(10) (implying $G = \sum_{\alpha=1}^{\infty} \frac{-1}{\mu_\alpha} Y_\alpha(\varphi) Y_\alpha(\tilde{\varphi})$) one has (corresponding to a vectorfield whose divergence is $\nabla_a f^a = -Y_\alpha$)

$$L_\alpha := \eta\zeta_\alpha := \int Y_\alpha \zeta\rho\, d^M\varphi = \frac{1}{\mu_\alpha} \int (\nabla^a Y_\alpha)\, \vec{p} \cdot \partial_a \vec{x} = \int f^a_\alpha\, \vec{p} \cdot \partial_a \vec{x}; \tag{24}$$

hence ($\approx$ indicating the use of (2) , i.e. equal modulo volume-preserving diffeomorphisms)

$$\begin{aligned}
&\mu_\alpha \mu_{\alpha'} \{\eta\zeta_\alpha, \eta\zeta_{\alpha'}\} \\
&= \left\{ \int \nabla^a Y_\alpha \vec{p} \cdot \partial_a \vec{x}\, d^M \varphi, \int \nabla^{a'} Y_{\alpha'} \vec{p} \cdot \partial_{a'} \vec{x}\, d^M \varphi' \right\} \\
&= \int \Big(\nabla^b Y_\alpha \nabla_b (\nabla^a Y_{\alpha'}) - \nabla^b Y_{\alpha'} \nabla_b (\nabla^a Y_\alpha) \Big)\, \vec{p} \cdot \partial_a \vec{x}\, d^M \varphi \\
&\approx -\int \Big(\nabla^b Y_\alpha \nabla_a \nabla_b \nabla^a Y_{\alpha'} - (\alpha \leftrightarrow \alpha') \Big)\, \eta\zeta\rho\, d^M \varphi, \qquad (25)
\end{aligned}$$

so that

$$f^\epsilon_{\alpha\alpha'} = \frac{\mu_{\alpha'} - \mu_\alpha}{2\mu_\alpha \mu_{\alpha'}} (\mu_\alpha + \mu_{\alpha'} - \mu_\epsilon) d_{\alpha\alpha'\epsilon}. \qquad (26)$$

For $M = 1$ the combination of eigenvalues gives $\frac{m^2 - n^2}{mn}$ which indeed (multiplying, in accordance with the conventional oscillator-expansions, the generators L_m by m) gives $(m - n)$.

Consequences of the dynamical symmetry, Lorentz-invariance in Matrix models, generalisations to the supersymmetric theories, and properties of the various algebras of local fields arising from $d_{\alpha\beta\gamma}$ and $e_{\alpha\beta\gamma}$ (cp. (21)) will be discussed in forthcoming papers.

Acknowledgement

I would like to thank M.Bordemann for innumerable discussions that, for many years, have always influenced my understanding.

References

[1] Jens Hoppe, *Quantum Theory of a Massless Relativistic Surface and a Two-dimensional Bound State Problem,* Ph.D. Thesis, MIT 1982, `http://dspace.mit.edu/handle/1721.1/15717` ;
see also *Membranes and Matrix Models*, `arXiv:hep-th/0206192`, IHES/P/02/47, and references therein.

[2] Jeffrey Goldstone, unpublished.